Coherentisation of First-Order Logic

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Preamble

Gentzen systems, both sequent calculus and natural deduction, are "natural", in the sense that proofs use logical constants in a simple (syntax-directed) way. By contrast, Hilbert systems and (from a different world) Resolution are "unnatural". We value naturality above efficiency.

An extension of Gentzen's ideas is the notion of "coherent" (aka "geometric") logic and the associated notion of "dynamical proof", as developed and popularised by various authors (Joyal, Reyes, Simpson, Negri, Lombardi, Bezem, Coquand). In such proofs, logical constants are invisible—their effect has disappeared into the notation.

Not all (first-order) theories are coherent. A folklore result from the 1970s shows that every theory has, by use of extra relation symbols, a coherent conservative relational extension, i.e. a theory can be "coherentised". Algorithms to do this tend to use a variant of "atomisation" (every formula becomes equivalent to an atom) or preprocessing to PNF (then CNF or DNF) or NNF, generating many new axioms.

We'll recall some of the history of this result and present a new coherentisation algorithm with the virtue of being "idempotent". We'll also discuss some of the relevant work on automation of coherent logic.

Background

Gentzen's calculi (e.g. **LK, LJ**): *logical axioms* (e.g. of form $A \wedge B \supset A$) are replaced by *inference rules* (e.g. $L\wedge$) acting on *sequents*.

Prolog: *Horn clauses* are in a very restricted form and usually implemented as rules operating on a single goal (rather than on a sequent).

Mathematics (esp. algebra): axioms are often of a simple form, such as universal formulae $\forall xy.\ xy = yx$, or "geometric", e.g. $\forall xy.P(x) \wedge P(y) \supset \exists z.L(z) \wedge I(x,z) \wedge I(y,z)$. (Name is both from connections with Grothendieck's algebraic geometry and from Euclid's geometry (and its successors).)

Negri's calculi: "mathematical axioms", e.g. $\forall xyz\ (Rxy \wedge Ryz) \supset Rxz$, are replaced by (schematic) inference rules acting on sequents, e.g.

$$\frac{Rxz, Rxy, Ryz, \Gamma \Rightarrow \Delta}{Rxy, Ryz, \Gamma \Rightarrow \Delta}\ Trans$$

provided the mathematical axioms are "geometric". (We are considering here her 2003 work, prior to the application to modal logics (2005).)

Reminder: Types of Skolemisation

Skolem (1928) introduced the technique now called *[Functional] Skolemisation*: it replaces existential quantifiers by function symbols, leading to a *"Skolem normal form"* (i.e. prefix of universal quantifiers, then body is quantifier-free). Old formula satisfiable iff new one is satisfiable. No suggestion that they are logically equivalent.

Dually, one may consider *Herbrandisation* (aka *Dual Functional Skolemisation*, replacing (in effect) universal quantifiers by function symbols, e.g. by negating, Skolemising as above and negating again, leading to a *dual Skolem normal form*. Old formula valid iff new one is valid. No suggestion that they are logically equivalent.

Skolem (1920) had an earlier (and, for us, more interesting) notion of *normal form*: $\forall$ then $\exists$ quantifiers, then quantifier-free. Achieved by adding fresh relation symbols. Function symbols (if any) left in place: no new ones. Old formula satisfiable iff new one is satisfiable. We call this Relational Skolemisation.

Coherent and Geometric Implications

A formula is positive, *aka "coherent"*, iff built from atoms (e.g. $\top, \bot, t = t', t \leqslant t', p(\mathbf{t}), \ldots$) using only $\vee$, $\wedge$ and $\exists$. Warning: model theorists also allow $\forall$.

A sentence is a coherent implication (CI) iff of the form $\forall \mathbf{x}.C \supset D$, where C, D are positive. [Neither coherent nor an implication ....]

A sentence is a special coherent implication (SCI) iff of the form $\forall \mathbf{x}.C \supset D$ where C is a conjunction of atoms and D is a finite disjunction of existentially quantified conjunctions of atoms.

Some restrict the notion of "coherent implication" to mean an SCI.

Old Theorem: Any coherent implication is intuitionistically equivalent to a finite conjunction of SCIs.

A formula is geometric iff built from atoms (as before) using only $\vee$, $\wedge$, $\exists$ and **infinitary** disjunctions. A sentence is a geometric implication iff of the form $\forall \mathbf{x}.C \supset D$, where C, D are geometric.

Similar terminology (SGI) and result for the infinitary (geometric) case.

Examples

Universal formulae $\forall \mathbf{x}.A$ (where A is quantifier-free) are equivalent to finite conjunctions of SCIs, just by putting A into CNF, distributing $\forall$ past $\wedge$ and rewriting (e.g. $\neg P \vee Q$ as $P \supset Q$). (No $\exists$ is involved. $\top$ and $\bot$ may be useful.)

Theory of *fields* is axiomatised by SCIs, including $\forall x.\top \supset (x = 0 \vee \exists y.xy = 1)$.

Theory of *real-closed fields* is axiomatised by countably many SCIs, including $\forall \mathbf{a}.\top \supset (a_{2n+1} = 0 \vee \exists x.a_{2n+1}x^{2n+1} + a_{2n}x^{2n} + \cdots + a_0 = 0)$.

Theory of *local rings* (rings with just one maximal ideal) is axiomatised by SCIs, including $\forall x.\top \supset (\exists y(xy = 1) \vee \exists y.(1 - x)y = 1)$.

Theory of *transitive relations* is axiomatised by SCI: $\forall xyz.(Rxy \wedge Ryz) \supset Rxz$.

Theory of *partial order* is axiomatised by SCIs, e.g. $\forall xy.(x \leqslant y \wedge y \leqslant x) \supset x = y$.

Theory of *strongly directed relations* is axiomatised by SCI: $\forall xyz.(Rxy \wedge Rxz) \supset \exists u.Ryu \wedge Rzu$.

(Infinitary) theory of *torsion abelian groups* is axiomatised by SGIs, including $\forall x.\top \supset \bigvee_{n>0}(nx = 0)$. [$nx$ stands for "sum of n copies of x"].

(Infinitary) theory of *fields of non-zero characteristic* is axiomatised by SGIs, including $\forall x.\top \supset \bigvee_{p>0}(px = 0)$.

Theory about Coherent Theories

A *coherent theory* is one axiomatised by [special] coherent implications. "Geometric" and "coherent" are often used synonymously.

1. *"Barr's Theorem":* Coherent implications form a "Glivenko Class", i.e. if a sequent $I_1, \ldots, I_n \Rightarrow I_0$ is classically provable, then it is intuitionistically provable, provided each I_i is a coherent implication. (Troelstra & van Dalen outline a short proof using Kripke models. Mints (2012) suggests a short proof ("due to Orevkov (1968)") using a multi-succedent intuitionistic sequent calculus; cf Loveland & Nadathur (1995). An instance of Barr's theorem (1974) in topos theory: Given a Grothendieck topos **E**, there is a complete Boolean algebra **B** and an exact cotripleable functor $E \to \mathcal{F}\mathbf{B}$).)

2. Coherent theories are those whose class of models is closed under filtered co-limits (calculated in **Set**) (Keisler (1960).

3. Coherent theories are "exactly the theories expressible by natural deduction rules in a certain simple form in which only atomic formulas play a critical part" (Simpson 1994). Similarly, SCIs can be converted directly to inference rules so that admissibility of the structural rules of the underlying sequent calculus is unaffected (Negri 2003).

G3c calculus of Ketone, Kleene et al

Γ, Δ are sets (or, if you prefer, multisets) of formulae; $\neg A =_{def} A \supset \bot$.

$$\frac{}{\bot, \Gamma \Rightarrow \Delta}\ L\bot \qquad \frac{}{P, \Gamma \Rightarrow \Delta, P}\ Ax$$

$$\frac{A, B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}\ L\wedge \qquad \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}\ R\wedge$$

$$\frac{A, \Gamma \Rightarrow \Delta \quad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta}\ L\vee \qquad \frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \vee B}\ R\vee$$

$$\frac{\Gamma \Rightarrow A, \Delta \quad B, \Gamma \Rightarrow \Delta}{A \supset B, \Gamma \Rightarrow \Delta}\ L\supset \qquad \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \supset B}\ R\supset$$

$$\frac{\forall x A, [t/x]A, \Gamma \Rightarrow A, \Delta}{\forall x A, \Gamma \Rightarrow \Delta}\ L\forall \qquad \frac{\Gamma \Rightarrow \Delta, [y/x]A}{\Gamma \Rightarrow \Delta, \forall x A}\ R\forall$$

$$\frac{\exists x A, [y/x]A, \Gamma \Rightarrow A, \Delta}{\exists x A, \Gamma \Rightarrow \Delta}\ L\exists \qquad \frac{\Gamma \Rightarrow \Delta, \exists x A, [t/x]A}{\Gamma \Rightarrow \Delta, \exists x A}\ R\exists$$

y is fresh in $R\forall$ and in $L\exists$, i.e. not free in the conclusion.

Conversion of Coherent Implications to Rules

When adding axioms to a first-order theory, one option for formalisation (in a cut-free sequent calculus such as **G3c**) is to include them all in the antecedents of all sequents.

But **if they are SCIs** it is more elegant to convert them to inference rules (Sequent Calculus: Negri 2003); (Natural Deduction: Simpson 1994).

We'll sometimes write an SCI without universal quantifiers; free variables are then schematic, i.e. instantiable as any terms we like.

Such an axiom $(P_1(\mathbf{x}) \wedge P_2(\mathbf{x}) \wedge \cdots \wedge P_n(\mathbf{x})) \supset D(\mathbf{x})$ is then converted to the rule

$$\frac{D(\mathbf{t}), P_1(\mathbf{t}), P_2(\mathbf{t}), \ldots, P_n(\mathbf{t}), \Gamma \Rightarrow \Delta}{P_1(\mathbf{t}), P_2(\mathbf{t}), \ldots, P_n(\mathbf{t}), \Gamma \Rightarrow \Delta}$$

in which the atoms $P_1(\mathbf{t}), P_2(\mathbf{t}), \ldots, P_n(\mathbf{t})$ are atoms in the conclusion's antecedent; the instance $D(\mathbf{t})$ can then (as we grow the derivation in a root-first fashion) be added to the antecedent.

Better still, analyse $D(\mathbf{t})$ immediately, using branching for analysis of $\vee$, fresh variables for analysis of $\exists$ and commas for analysis of $\wedge$.

Conversion of Coherent Implications to Rules: Examples

We'll use variables rather than terms, but understand them (unless fresh) to be instantiable as terms.

- The coherent implication $\forall xyz.(x \leqslant y \wedge y \leqslant z) \supset (y \leqslant x \vee z \leqslant y)$ is converted to the inference rule

$$\frac{y \leqslant x, x \leqslant y, y \leqslant z, \Gamma \Rightarrow \Delta \qquad z \leqslant y, x \leqslant y, y \leqslant z, \Gamma \Rightarrow \Delta}{x \leqslant y, y \leqslant z, \Gamma \Rightarrow \Delta}$$

  (If $\leqslant$ is a partial order, this says that the depth is at most 2.)

- The coherent implication $\forall xyz.(x \leqslant y \wedge x \leqslant z) \supset \exists w(y \leqslant w \wedge z \leqslant w)$ is converted to the inference rule (in which w is fresh, i.e. not in the conclusion):

$$\frac{y \leqslant w, z \leqslant w, x \leqslant y, x \leqslant z, \Gamma \Rightarrow \Delta}{x \leqslant y, x \leqslant z, \Gamma \Rightarrow \Delta}$$

  (If $\leqslant$ is a partial order, this says that it is "strongly directed".)

Conversion to Rules and Q-style Focusing

Conversion of an SCI $\forall \mathbf{x}. C \supset D$ to a rule exploits the relatively simple form (conjunction C of atoms) of the antecedent of an SCI.

In root-first proof search, in a **G3c**-style sequent calculus, to use such an SCI in the antecedent of a sequent, one may, rather than

- setting the sub-formula C (suitably instantiated) as a new goal and
- branching with the sub-formula D (suitably instantiated) as a new assumption,

just wait until the atoms making up C are already (in instantiated form) in the sequent's antecedent.

So a two-premiss rule becomes a one-premiss rule (before we consider the effects of disjunctions in D).

We call this "Q-style focusing" since similar phenomena appear in the use of the calculi **LJQ** and **LKQ** of Herbelin (1995).

As observed by (e.g.) Polonsky, it is also much the same idea as in **hypertableaux** (and thus also in **hyperresolution**).

Brief Notes on Q-style Focusing

LJQ' is a single succedent sequent calculus for **Int**, with *unfocused* sequents $\Gamma \Rightarrow A$ and *focused* sequents $\Gamma \rightarrow A$, where Γ is a multiset and A a formula. The *focus* is the formula A. See RD & Lengrand (2006).

At any stage (in root-first search) one can switch from a non-focused sequent to a focused sequent; the reverse switch is not allowed, save that the right rules for implication and universal quantification have a focused conclusion but a non-focused premiss. Left rules have unfocused conclusions.

The other right rules (for existential quantification, conjunction and disjunction) have focused premisses; initial sequents $\Gamma, P \rightarrow P$ are focused and, the left rule for implication has as its first premiss the focused sequent $\Gamma, A \supset B \rightarrow A$. **Note that if A is atomic then this is derivable iff $A \in \Gamma$; similarly for a conjunction of atoms (all must be in Γ).**

Cut-admissibility, and thus completeness, of this calculus is routine.

There's also a multi-succedent version, and a classical version. The "Q" is from "Queue" (tail) as opposed to "T" for "Tête" (head).

As consequences, one has easy proofs of the completeness of Dragalin's **GHPC**, of the depth-bounded calculus **G4ip**, and (using the same ideas) of calculi for guarded logic (where an atomic *key* unlocks the guard).

Naturality, Applicability and Efficiency

Naturality: the inference rules generated from SCIs are logic-free, and cover several logical steps at once. Proofs using them are clear and transparent compared to resolution or even functionally Skolemised alternatives. Some examples later.

Applicability: we've given some examples showing that some (many?) mathematical theories have coherent implications as axioms. We'll show later that **every** first-order theory can (using a conservative relational extension) be put into this form (a little-known folklore result from the 1970s); our novel contribution is an improved algorithm for doing this.

Efficiency: We'll consider this in the context of automation. Naturality has obvious positive consequences for efficiency of manual usage.

Automation

Bezem & Coquand (2005), de Nivelle and Meng (2006), Berghofer (2008), Polonsky (2010), Stojanović et al (2011), Fisher (2012) and de Nivelle (2014) have experimented with automation of coherent logic. We wish that proofs should be natural rather than based on pre-processing to a NF. We might also care that our proofs should be constructive, since "Barr's Theorem" tells us that they should be (except where the goal is arbitrary—case 6 below).

Let the context Θ be a coherent theory. We can consider

1. Proving absurdity $\bot$.
2. Proving an atomic sentence $P(\mathbf{t})$.
3. Proving a conjunction $C_i(\mathbf{t})$ of atomic sentences $P_{ij}(\mathbf{t})$.
4. Proving a [special] positive sentence $D = \vee_{i=1}^{n} \exists \mathbf{y}. C_i(\mathbf{y})$.
5. Proving an [S]CI $\forall \mathbf{x}.\ C(\mathbf{x}) \supset D(\mathbf{x})$.
6. Proving an arbitrary sentence A.
7. Seeing what ground atoms follow by Θ from some set of such atoms.
8. ...

Automation 2

Problem 3 (conjunctions of atoms) is easily reduced to several instances of problem 2 (one atom).

With almost no loss of naturality, we can replace a goal $P(\mathbf{t})$ by $\bot$ and add the sentence $P(\mathbf{t}) \supset \bot$ to the theory, thus reducing problem 2 to 1. Because of the focused nature of the rules, the rule associated with the new sentence $P(\mathbf{t}) \supset \bot$ will only fire once $P(\mathbf{t})$ has been achieved, so this use of double negation is only superficial—constructivity is not lost.

Problem 4, for positive D (other than $\bot$), can be solved (by adding $\neg D$ to Θ) by "uniform proof" techniques [Nadathur & Loveland (1995)].

Problem 5, for the [S]CI $\forall \mathbf{x}.C \supset D$, is easily reduced to problem 4, by adding $C(\mathbf{c})$ to the theory and proving $D(\mathbf{c})$, with new constants $\mathbf{c}$.

Problem 6 (goal is arbitrary) can (with loss of naturality) be reduced to problem 1, by negating the formula, coherentising it (see below), adding the results to the theory and proving $\bot$. [Bezem & Coquand (2005)]

But what interests us is something more like Problem 7 (or 8).

Automation 3

In contrast to resolution, there are no "Skolem function symbols", thanks to the restricted nature of the syntax; so the Herbrand universe is more limited (typically, finite).

Bezem & Coquand (2005) illustrate the ideas with a case study, the preservation of a relation's diamond property under reflexive closure.

Bezem & Coquand (2003) illustrates, with a proof of the inductive step in Newman's Lemma, that coherent reasoning can be "orders of magnitude" faster than resolution.

Bezem & Hendriks (2008) give a (coherent) proof of Hessenberg's theorem (Pappus planes are Arguesian) from projective geometry, not then achieved by any other automated theorem prover. (This proof has been verified in *Coq*; there is also a 2008 proof in *Coq*, by Magaud et al, not using coherent logic.)

Because of the constructive nature of coherent logic, all proofs can be converted to natural deduction proofs and lambda-terms extracted as algorithms (and checked by *Coq*).

Automation 4: Problems of Type 8

For our application (details later) to labelled sequent calculi, we are not so much interested in seeing whether a theory $\mathcal{T}$ proves a certain sentence, or even $\bot$, but in exploring the atomic consequences, w.r.t. a theory $\mathcal{T}$, of a set Θ (maybe called a "database" or a "model") of atoms.

One way of expressing this is as the enumeration of all solutions (i.e. instantiations of $\mathbf{x}$) to the "query" $R(y, \mathbf{x})$, where R is a relation symbol of interest and y is a known label, e.g. of a modal formula.

"The Chase" is an algorithm [Maier et al (1979)] for this from database science. $\mathcal{T}$ might be a "set of tuple-generating dependencies" (aka "TGDs") or even a "set of disjunctive existential rules" (aka "DTGDs").

Ideally such rules are "guarded", i.e. all universally quantified variables are "guarded" by one or more atoms. (SCIs like $\forall x.\top \supset R(x, x)$ are not guarded; addition of a unary predicate *dom* can fix this.)

But our problem is yet more complex: generation of answers to queries is **interleaved** with generation of fresh labels y and fresh atoms $R(x, y)$.

Other Kinds of Sentences and Theories

Not all sentences, and not all theories, are coherent implications, or even equivalent to (resp. axiomatised by) a set of such things.

- The "McKinsey condition" (a frame condition for modal logic, related to the McKinsey axiom $\Box\Diamond A \supset \Diamond\Box A$)

  $$\forall x \exists y.\ xRy \wedge (\forall z.\ yRz \supset y = z)$$

  is not a coherent implication. [We can't shift the $\forall z$ out past $\exists y$.]
- The "strict seriality condition"

  $$\forall x.\ \exists y.\ xRy \wedge \neg(yRx)$$

  is likewise not a coherent implication, because of the negation.
- The axioms for Henselian local rings provide another example of an axiom that is almost a coherent implication.
- The frame condition for Kreisel-Putnam logic (see later) is almost a coherent implication.

Solutions

The technique we adopt is an adaptation (and extension) of Skolem's method (1920) of what we call "Relational Skolemisation".

- Introduce a new unary predicate symbol M (for *Maximal*, with $M(y)$ "meaning" $\forall z.\ yRz \supset y = z$), and replace the McKinsey condition $\forall x \exists y.\ xRy \wedge (\forall z.\ yRz \supset y = z)$ by two SCIs:

$$\forall yz.\ (M(y) \wedge yRz) \supset y = z$$

$$\forall x.\ \top \supset (\exists y.\ xRy \wedge M(y))$$

- Introduce a new binary predicate symbol S, with xSy "meaning" $\neg xRy$ and replace the strict seriality condition $\forall x.\exists y.\ xRy \wedge \neg(yRx)$ by two SCIs:

$$\forall xy.\ (xRy \wedge xSy) \supset \bot$$

$$\forall x.\top \supset \exists y.xRy \wedge ySx$$

What is going on here? Why is "meaning" in quotes?

Reminder

A theory $\mathcal{T}'$ in the language $\mathcal{L}'$ is a model-theoretic conservative extension of $\mathcal{T}$ in $\mathcal{L}$ iff

(i) $\mathcal{L} \subseteq \mathcal{L}'$;

(ii) $\mathcal{T} \subseteq \mathcal{T}'$;

(iii) the symbols of $\mathcal{L}' \backslash \mathcal{L}$ can be interpreted in some fashion so that every model $\mathcal{M}$ of $\mathcal{T}$ is a model $\mathcal{M}'$ of $\mathcal{T}'$ (where the symbols of $\mathcal{L}$ are interpreted just as $\mathcal{M}$ interprets them).

If the "extension" of $\mathcal{M}$ to $\mathcal{M}'$ can be done in just one way, one talks of *strong model-theoretic conservativity*.

Such an extension is (easily) a conservative extension, and some of the treatments of our subject in the literature take care to construct such an extension, or even a strong one. But, that is more than **we** need!

A Solution Not Adopted Here

To construct a coherent conservative extension of a f.o.-theory $\mathcal{T}$:

For each axiom, apply Functional Skolemisation, thus constructing a $\forall$-sentence. Its body can be put into CNF. Move conjunctions outwards; so one has a number of universally quantified clauses. Each clause can be written as an implication from a conjunction of atoms to a disjunction of atoms, i.e., as a simple form of SCI.

This is of course very very familiar, well-known to give a set of sentences that are collectively satisfiable iff the original axiom is satisfiable, and not (in general) to give an "equivalent" formula.

Despite the latter negative comment, it does give a model-theoretic conservative extension $\mathcal{T}'$ of $\mathcal{T}$; see Sec 3.4 of van Dalen's book "Logic & Structure". Hence the (proof-theoretic) conservativity of $\mathcal{T}'$ over $\mathcal{T}$.

But, the introduction of function symbols, and use of CNF, however, not merely makes the translation opaque but also hinders the automation, if we follow the B&C argument about the size of the Herbrand universe.

Results

By a *Skolem extension* of a theory $\mathcal{T}$ we mean an extension by new relation symbols and replacement of old axioms by new axioms such that (i) every old axiom is still provable and (ii) there is a substitution of formulae in the old language for the new symbols so that all the new axioms become theorems of $\mathcal{T}$.

Theorem If $\mathcal{T}'$ is a Skolem extension of $\mathcal{T}$, then (i) it is a conservative extension and (ii) they are satisfiable in the same domains (and hence are equi-satisfiable).

Proof Routine. □

Old Theorem [Skolem (1920)] Every first-order theory has a Skolem extension axiomatised by $\forall\exists$-sentences (one axiom for each old axiom).

Proof Generalisation of just the first trick given above. One method is to replace each axiom by its prenex normal form, and replace any quantifier alternation $\forall\exists\forall$ by just $\forall$ together with a new relation symbol R and a new $\forall\exists$-sentence giving the meaning of R. The remains of the old axiom and the new axioms can even be conjoined, to give a single $\forall\exists$-sentence. (This converts a formula to its *Skolem Normal Form*, a *Relational Skolemisation* (one of many).)

Results 2

Theorem [???] Every first-order theory has a Skolem extension axiomatised by [special] coherent implications.

Proof Generalisation of the two tricks given above. One method is to replace each axiom by its prenex normal form, with the body in CNF or in DNF, and use one trick to strip off pairs $\forall \mathbf{x} \exists \mathbf{y}$ of quantifiers and the other trick to get rid of negated atoms. New relation symbols are introduced, along with special coherent implications that (partially) express their meanings. [See below, or our paper, for exact details.] □

Corollary Every first-order theory has a conservative relational extension axiomatised by [special] coherent implications.

Corollary Every first-order theory has a relational $\forall\exists$-extension (i.e. an extension axiomatised by $\forall\exists$ formulae) with models in the same domains.

Remark This corollary is the result as stated and proved by Skolem 1920, as a means to simplify the proofs of Löwenheim's theorems about cardinality of models. His proof achieves something stronger: the result we gave above about Skolem Normal Form.

Caution

It may appear that all we are doing is constructing a definitional extension of the language.

Here is a counter-argument. Let us first distinguish **three** ways of extending a theory with (for example) a new unary symbol M, informally with $M(y)$ "meaning" $\forall z.y \leqslant z \supset y = z$:

- Adding an abbreviative definition (aka "[definitional] abbreviation"). The new symbol is a "defined symbol", so the formula $M(y)$ is indistinguishable from the formula $\forall z.y \leqslant z \supset y = z$; in particular, it is neither atomic nor even quantifier-free.
- Making a definitional extension (aka an "extension by definitions"), i.e. adding a fresh primitive relation symbol M and a new axiom $\forall y.\ M(y) \equiv (\forall z.\ y \leqslant z \supset y = z)$. Well-known to give a conservative extension [Shoenfield (1967), van Dalen's book].
- Making a semi-definitional extension, i.e. adding a fresh primitive relation symbol M and a new axiom: $\forall y.M(y) \supset \forall z.y \leqslant z \supset y = z$ [or the equivalent SCI $\forall yz.M(y) \wedge y \leqslant z \supset y = z$]. This suffices for our needs. Note that $\forall y.\ (\forall z.y \leqslant z \supset y = z) \supset M(y)$ is not a CI.

Caution, 2

The theory of posets in which **every element is below a maximum element** is axiomatised by easy conditions including the McKinsey Condition: $\forall x.\, \exists y.\, x \leqslant y \wedge \forall z. y \leqslant z \supset y = z$.

Model-theoretic considerations show the theory to have no coherent axiomatisation. (Its models (with order-preserving morphisms) are not closed under filtered direct limits in **Set**.)

It doesn't help to make the **abbreviations** $M(y) \equiv \forall z(y \leqslant z \supset y = z)$ and $N(x, y) \equiv \neg(x = y)$; this doesn't change the classes of models and of their morphisms. [See next slide for why N is useful.]

But the other techniques described construct a conservative extension that **is** coherent. (Having a new **primitive** M **changes** the class of morphisms.)

There is thus a subtle difference between the other techniques described and the use of abbreviations; the latter can't change an incoherent theory into a coherent one. For details of the argument, based on Johnstone (2002), see our paper.

Caution, 3

But how does what we do differ from Shoenfield's "extension by definitions"?

We could add both halves of the equivalence, i.e. **both** $\forall yz. M(y) \wedge y \leqslant z \supset y = z)$ **and** $\forall y. (\forall z. y \leqslant z \supset y = z) \supset M(y)$; that gives an extension by definitions.

The first formula is an SCI; the second formula is not even a CI. Rearranging it classically, we get $\forall y.\ M(y) \vee \exists z. y \leqslant z \wedge y \neq z$.

A fresh relation symbol N and the axiom $\forall yz.\ N(y, z) \equiv y \neq z$ allows the second formula to be turned into an SCI ; the new axiom is equivalent to the conjunction of $\forall yz.\ (N(y, z) \wedge y = z) \supset \bot$ and $\forall yz.\ y = z \vee N(y, z)$.

Two fresh relation symbols and three SCIs in total, and effectively a definitional extension.

But, we don't need the second formula, provided that we are only replacing a **positive** occurrence of an instance of $\forall z. y \leqslant z \supset y = z)$.

We have therefore introduced the notion of a semi-definitional extension.

Labelled Calculi for Modal and Intuitionistic Logics

One way (Negri 2005) to formalise modal logics is to take a classical sequent calculus **G3c** (with all inference rules invertible), and label each formula (labels are simple, with no structure).

The rules for the modal operators are special, using extra formulae xRy relating the labels x and y. Kripke semantics is thus internalised. This is a sequent calculus version of the much older approach of "prefixed tableaux".

Unsurprisingly, something similar (RD & Negri 2012) can be done for intuitionistic logic (and intermediate logics), with monotonicity at atoms and special rules for implication using a quasi-order $\leqslant$ as the accessibility (aka "frame") relation.

We'll focus on this rather than on modal logic.

Labelled Calculi for Modal and Intuitionistic Logics, 2

Rules for implication are then:

$$\frac{x \leqslant y, x : A \to B, \Gamma \Rightarrow y : A, \Delta \quad x \leqslant y, x : A \to B, y : B, \Gamma \Rightarrow \Delta}{x \leqslant y, x : A \to B, \Gamma \Rightarrow \Delta} \; L{\to}$$

and, with y fresh (i.e. not in the conclusion),

$$\frac{x \leqslant y, y : A, \Gamma \Rightarrow y : B, \Delta}{\Gamma \Rightarrow x : A \to B, \Delta} \; R{\to}$$

and also reflexivity (for x in the conclusion) and transitivity:

$$\frac{x \leqslant x, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \; \mathit{Refl} \quad \frac{x \leqslant z, x \leqslant y, y \leqslant z, \Gamma \Rightarrow \Delta}{x \leqslant y, y \leqslant z, \Gamma \Rightarrow \Delta} \; \mathit{Trans}.$$

Labelled Calculi for Intermediate Logics

Intermediate (aka "superintuitionistic") logics are those, e.g. Gödel -Dummett logic and Jankov-De Morgan logic, between intuitionistic and classical logic.

They can usually be presented using some "frame conditions", e.g. $\forall xyz.\ (x \leqslant y) \wedge (x \leqslant z) \supset \exists w.(y \leqslant w) \wedge (z \leqslant w)$ for J-DM logic and $\forall xy.(x \leqslant y) \vee (y \leqslant x)$ for G-D logic; so we need to incorporate such conditions into the rules of the sequent calculus.

Where (as here) the condition is an SCI, this is easy:

$$\frac{xl \leqslant y, x \leqslant z, y \leqslant w, z \leqslant w, \Gamma \Rightarrow \Delta}{x \leqslant y, x \leqslant z, \Gamma \Rightarrow \Delta}\ J - DM$$

$$\frac{x \leqslant y, \Gamma \Rightarrow \Delta \quad y \leqslant x, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ G - D$$

(we can restrict $G - D$ to cases where x and y are in the conclusion):

Labelled Calculi for Intermediate Logics 2

Not all frame conditions are SCIs. E.g., (i) the McKinsey condition (in modal logic) above and (ii) that for the Kreisel-Putnam (intermediate) logic, axiomatised by $(\neg A \to (B \vee C)) \to ((\neg A \to B) \vee (\neg A \to C))$.

The condition for (ii) is

$$\forall xyz.\ (x \leqslant y \ \wedge\ x \leqslant z) \ \supset\ (y \leqslant z \ \vee\ z \leqslant y \ \vee\ \exists u.(x \leqslant u \ \wedge\ u \leqslant y \ \wedge\ u \leqslant z \ \wedge\ F(u, y, z)))$$

where $F(u, y, z)$ abbreviates $\forall v.\ u \leqslant v \supset \exists w.\ (v \leqslant w \wedge (y \leqslant w \vee z \leqslant w))$;

By changing F from a (bi-directional) abbreviation to a new predicate symbol with an associated SCI (in just one direction)

$$\forall uvyz.\ (F(u, y, z) \ \wedge\ u \leqslant v) \supset (\exists w(v \leqslant w \ \wedge\ y \leqslant w) \vee \exists w(v \leqslant w \ \wedge\ z \leqslant w))$$

we achieve our goal of making the condition for (ii) an SCI. The theory developed above about conservative extensions formalises this idea.

So, the conversion of non-coherent frame conditions to SCIs allows the application of coherent logic to a wider range of modal and intermediate logics.

Rival Approaches

Negri's approach (labelled sequent calculi, with geometric rules) is just one of several for going beyond the formalism or even the expressive power of traditional Gentzen systems for non-classical logics:

Nested sequent calculi and hypersequent calculi. But these are (as shown by Fitting in the former case) subsumed by the labelled approach, provided the full power of FOL is available.

Representation of frame conditions as inference rules:

1. For arbitrary frame conditions from FOL, one can use functional Skolemisation to obtain ∀-sentences, then put the body into CNF, then generate rules that act on both sides of the sequent [Castellini & Smaill (2002), Schmidt & Tishkovksy (2011)];
2. Where the frame conditions are ∀∃-sentences, one can allow use of both sides of sequents [Ciabattoni, Maffezioli and Spendier (2013)];
3. Where the frame conditions are SCIs, we represent them directly as inference rules (on just the LHS) [Negri (2005); RD & Negri (2012)]).

Rival Approaches: Commentary

Some (e.g. Ciabattoni et al (2013)) have suggested that using the full power of FOL (or even just $\forall\exists$-sentences) might allow more logics than the restriction to coherent logic allows.

No: Relational Skolemisation (just as in Skolem (1920)) applies to "replace" any FOL sentence by a $\forall\exists$-sentence, with a conservative extension result; so it allows use of the approach of Ciabattoni et al. (The **stronger** result, constructing SCIs, allows the use of Negri's approach.)

SCI-generated rules are natural and transparent, whereas both functional Skolemisation (with clausification) and the 1920-style relational Skolemisation tends to render the conditions rather opaquely. Considering SCIs as $\forall\exists$-sentences is opaque: contrast

$$\forall xyz.\ (x \leqslant y) \wedge (x \leqslant z) \supset \exists w.(y \leqslant w) \wedge (z \leqslant w),$$

where w is only constructed if $(x \leqslant y) \wedge (x \leqslant z)$, and its f.o.-equivalent in $\forall\exists$-form

$$\forall xyz \exists w.\ (x \leqslant y) \wedge (x \leqslant z) \supset (y \leqslant w) \wedge (z \leqslant w).$$

Rival Approaches: Example

In the approach of Castellini & Smaill, the frame condition

$$\forall x \exists y.\ x \prec y \land (\forall z. y \prec z \supset y = z)$$

generates the rule

$$\frac{\Gamma, \tau_1 \prec la(\tau_1), la(\tau_1) = \tau_2 \ \Rightarrow\ \Delta \qquad \Gamma, \tau_1 \prec la(\tau_1) \ \Rightarrow\ la(\tau_1) \prec \tau_2, \Delta}{\Gamma \Rightarrow \Delta}\ atom$$

where la is a Skolem function symbol. **Are you lost yet**?

In their root-first proof using this of the McKinsey sequent $0 : \Box\Diamond A \Rightarrow 0 : \Diamond\Box A$, the rule $R\Diamond$ is activated not by an antecedent formula $0 \prec y$ but by the possibility, to be justified in a new branch, of finding some term τ (in fact $la(0)$) for which $0 \prec \tau$ can be proved.

In our approach, the condition is converted to the SCIs $\forall x. \top \supset \exists y. x \prec y \land My$ and $\forall yz.(My \land y \prec z) \supset y = z$ and thus to the two schematic rules (with y fresh in the first)

$$\frac{x \prec y, My, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \gamma \qquad \frac{My, y \prec z, y = z, \Gamma \Rightarrow \Delta}{My, y \prec z, \Gamma \Rightarrow \Delta}\ \delta$$

Rival Approaches: Example (continued)

The rule γ corresponding to the first of these implications has $x \prec y$ and My in the premiss but not in the conclusion; but its use is triggered only by an instance of x in the conclusion and the number of possible new formulae of this kind is finite, whereas presence of Skolem function symbols f would allow arbitrarily many instantiations $f(x), f(f(x)), \ldots$.

Our proof is then

$$\overline{\boxed{y : A}, \ldots \Rightarrow \boxed{y : A}, \ldots} \; Ax$$

$$\overline{\boxed{y = z}, \boxed{y = w}, \ldots, \boxed{w : A}, \ldots \Rightarrow \boxed{z : A}, \ldots} \; =, =$$

$$\overline{\boxed{y \prec z}, \boxed{y \prec w}, x \prec y, \boxed{My}, w : A, y : \Diamond A, \ldots \Rightarrow z : A, y : \Box A, \ldots} \; \delta, \delta$$

$$\overline{x \prec y, My, \boxed{y : \Diamond A}, \ldots \Rightarrow \boxed{y : \Box A}, \ldots} \; R\Box, L\Diamond$$

$$\overline{x \prec y, My, \boxed{x : \Box\Diamond A} \Rightarrow \boxed{x : \Diamond\Box A}} \; L\Box, R\Diamond$$

$$\overline{x : \Box\Diamond A \Rightarrow x : \Diamond\Box A} \; \gamma$$

$$\Rightarrow \boxed{x : \Box\Diamond A \supset \Diamond\Box A} \; R\supset$$

.

Example Proof from Castellini & Smaill

$$\dfrac{\dfrac{\dfrac{}{p \mathbin{@} t_2 \longrightarrow p \mathbin{@} t_2}\,\textsf{ax}}{t_1 \doteq t_2, p \mathbin{@} t_1 \longrightarrow p \mathbin{@} t_2}\,\textsf{sub}_{\doteq}}{la(0) \doteq t_1, la(0) \doteq t_2, p \mathbin{@} t_1 \longrightarrow p \mathbin{@} t_2}\,\textsf{sub}_{\doteq}$$

Branch $\boxed{3}$

$$\dfrac{\dfrac{}{0 \prec la(0) \longrightarrow 0 \prec la(0)}\,\textsf{ax} \qquad \dfrac{}{0 \prec la(0) \longrightarrow 0 \prec la(0)}\,\textsf{ax}}{\longrightarrow 0 \prec la(0)}\,\textsf{atom}$$

Branch $\boxed{2}$

$$\dfrac{\dfrac{}{0 \prec la(0) \longrightarrow 0 \prec la(0)}\,\textsf{ax} \qquad \dfrac{}{0 \prec la(0) \longrightarrow 0 \prec la(0)}\,\textsf{ax}}{\longrightarrow 0 \prec la(0)}\,\textsf{atom}$$

Branch $\boxed{1}$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\boxed{3} \quad \dfrac{}{la(0) \prec t_1 \longrightarrow la(0) \prec t_1}\,\textsf{ax}}{la(0) \prec t_1, la(0) \doteq t_1, p \mathbin{@} t_2 \longrightarrow p \mathbin{@} t_1}\,\textsf{atom} \qquad \dfrac{}{la(0) \prec t_2 \longrightarrow la(0) \prec t_2}\,\textsf{ax}}{la(0) \prec t_1, la(0) \prec t_2, p \mathbin{@} t_2 \longrightarrow p \mathbin{@} t_1}\,\textsf{atom}}{la(0) \prec t_1, \Diamond p \mathbin{@} la(0) \longrightarrow p \mathbin{@} t_1}\,l\Diamond \qquad \boxed{2}}{la(0) \prec t_1, \Box\Diamond p \mathbin{@} 0 \longrightarrow p \mathbin{@} t_1}\,l\Box}{\Box\Diamond p \mathbin{@} 0 \longrightarrow \Box p \mathbin{@} la(0)}\,r\Box \qquad \boxed{1}}{\Box\Diamond p \mathbin{@} 0 \longrightarrow \Diamond\Box p \mathbin{@} 0}\,r\Diamond}{\longrightarrow \Box\Diamond p \supset \Diamond\Box p \mathbin{@} 0}\,r\supset$$

Bottom of the tree

Prior and Related Work

Was such a simple result (the coherent conservative relational extension theorem) really not known before ?

- Skolem (1920) showed how to replace a f.-o. formula by a single ∀∃-formula "satisfiable in the same domains".
- Antonius (1975) used a similar technique to replace a f.-o. formula by a positive formula (plus lots of coherent implications), with a theorem slightly weaker than what we need.
- Johnstone (2002) expresses a similar result in terms of models in Boolean coherent categories. He learnt it (he says) from Sacks' Chicago lectures (1975–76), but knows of no pre-2002 publication.
- Sacks (2015, pers. corr.) says that the ideas are "new" to him.
- Bezem & Coquand (2005) use another similar technique: their result is just about satisfiability (the original formula is unsatisfiable iff its coherentisation—lots of SCIs—is unsatisfiable). Not quite as strong as what we need. Mints (2012) is similar. A conservativity result can be proved using their translation.
- Shulman (2012) says "Every finitary first-order theory is equivalent, over classical logic, to a coherent theory" (with references: one inaccurate and one unhelpful).

Summary about Priority

So, to whom do we attribute the first statement and proof of the result that every f.o.-theory has a coherent conservative relational extension?

I see no such result in Skolem (1920) or (1928).

Wedad Antonius was a student of Gonzalo Reyes in Montreal. She is now a teacher. Her "thèse de maitrise" is unpublished; by chance I obtained a copy, and have been in touch with her. I omit her exact result and construction for lack of time.

Her result is slightly weaker; it is summarised by Reyes (1977) as

> *Every classical theory may be rendered coherent by extending the language.*

and by Marquis and Reyes (2012) as

> *Any classical theory can be translated in[to] the language of coherent logic provided that the latter is enriched with sufficiently many relational symbols.*

which I accept as evidence for her priority.

Johnstone's Statement of the Result

Johnstone (2002, Lemma D 1.5.13, p 858) is worth reporting *verbatim*:

> "Let $\mathcal{T}$ be a first-order theory over a signature Σ. Then there is a signature Σ' containing Σ, and a coherent theory $\mathcal{T}'$ over Σ', such that for any Boolean coherent category $\mathcal{C}$ we have $\mathcal{T}\text{–}\mathbf{Mod}(\mathcal{C})_e \simeq \mathcal{T}'\text{–}\mathbf{Mod}(\mathcal{C})$.
> *The theory $\mathcal{T}'$ is sometimes called the* Morleyization *of $\mathcal{T}$, in honour of M. Morley (cf. Hodges (1993)).*"

The (easily missed) suffix e refers to the restriction to elementary morphisms within the category of models. For our present purposes, $\mathcal{C}$ may be taken to be the category of sets. The reference is to Hodges' book on model theory, which disavows the name "Morleyization" in favour of "atomisation" (on the grounds that it was nothing to do with Morley); see also Blass' comment (on Stackexchange in 2002) that "this name for such a triviality could be considered an insult to Michael Morley". Nevertheless, from this Lemma one can extract a (strong model-theoretic) conservativity result.

"Atomisation" refers to the technique of proof: every formula of the language is made to be equivalent to an atom, so (very easily) there are lots of very simple SCIs.

Existing Coherentisation Algorithms: 1

Bezem & Coquand (2005) show as follows how to atomise, i.e. how to replace an axiom ϕ of a theory $\mathcal{T}$ by new axioms in SCI form with the property that $\mathcal{T}$ is consistent iff the new theory $\mathcal{T}'$ is consistent:

For each subformula $\psi(\boldsymbol{x})$ of ϕ we introduce two atomic predicates $T(\psi)(\boldsymbol{x})$ and $F(\psi)(\boldsymbol{x})$ with the following coherent axioms

$$\text{if } \psi(\boldsymbol{x}) \text{ is } \psi_1 \wedge \psi_2 \text{ then } \begin{cases} T(\psi)(\boldsymbol{x}) \to T(\psi_1)(\boldsymbol{x}) \wedge T(\psi_2)(\boldsymbol{x}) \\ F(\psi)(\boldsymbol{x}) \to F(\psi_1)(\boldsymbol{x}) \vee F(\psi_2)(\boldsymbol{x}) \end{cases}$$

$$\text{if } \psi(\boldsymbol{x}) \text{ is } \psi_1 \vee \psi_2 \text{ then } \begin{cases} T(\psi)(\boldsymbol{x}) \to T(\psi_1)(\boldsymbol{x}) \vee T(\psi_2)(\boldsymbol{x}) \\ F(\psi)(\boldsymbol{x}) \to F(\psi_1)(\boldsymbol{x}) \wedge F(\psi_2)(\boldsymbol{x}) \end{cases}$$

$$\text{if } \psi(\boldsymbol{x}) \text{ is } \psi_1 \to \psi_2 \text{ then } \begin{cases} T(\psi)(\boldsymbol{x}) \to F(\psi_1)(\boldsymbol{x}) \vee T(\psi_2)(\boldsymbol{x}) \\ F(\psi)(\boldsymbol{x}) \to T(\psi_1)(\boldsymbol{x}) \wedge F(\psi_2)(\boldsymbol{x}) \end{cases}$$

$$\text{if } \psi(\boldsymbol{x}) \text{ is } \neg\psi_1 \text{ then } \begin{cases} T(\psi)(\boldsymbol{x}) \to F(\psi_1)(\boldsymbol{x}) \\ F(\psi)(\boldsymbol{x}) \to T(\psi_1)(\boldsymbol{x}) \end{cases}$$

$$\text{if } \psi(\boldsymbol{x}) \text{ is } \forall y.\psi_1(\boldsymbol{x}, y) \text{ then } \begin{cases} T(\psi)(\boldsymbol{x}) \to T(\psi_1)(\boldsymbol{x}, y) \\ F(\psi)(\boldsymbol{x}) \to \exists x.F(\psi_1)(\boldsymbol{x}, y) \end{cases}$$

$$\text{if } \psi(\boldsymbol{x}) \text{ is } \exists y.\psi_1(\boldsymbol{x}, y) \text{ then } \begin{cases} T(\psi)(\boldsymbol{x}) \to \exists x.T(\psi_1)(\boldsymbol{x}, y) \\ F(\psi)(\boldsymbol{x}) \to F(\psi_1)(\boldsymbol{x}, y) \end{cases}$$

if $\psi(\boldsymbol{x})$ is atomic then $T(\psi)(\boldsymbol{x}) \to \neg F(\psi)(\boldsymbol{x})$ (or: $T(\psi)(\boldsymbol{x}) \wedge F(\psi)(\boldsymbol{x}) \to \bot$)

Existing Coherentisation Algorithms: 2

This method generates a **lot** of new relation symbols: it does however almost merit the name of "atomisation", i.e. every subformula of each axiom is now equivalent to an atom. Half of the new axioms can be dispensed with by consideration of polarities. The approach actually generates two new relation symbols per formula rather than just one.

The technique is from Johnstone (2002), which applies it not just to axioms but to **all** formulae of the language (now almost meriting the name of "atomisation").

Antonius' method (1975) is more subtle, generating far fewer new relation symbols.

Another approach is just to modify Skolem's 1920 argument, using PNF and then either DNF or CNF to normalise the body. Or one can put the formula into NNF.

For more details of these approaches, with comparison of worked examples, see our paper (2015).

New Coherentisation Algorithm

Existing published algorithms to convert a sentence into a finite set of SCIs may not just generate lots of new relation symbols, but also fail to be *idempotent*, i.e. to leave an SCI unchanged, since both atomisation and conversion to PNF (and then to CNF or DNF, or to NNF) can destroy too much of the sentence's structure. Can we do any better?

Definition [RD & SN]. A formula is *weakly positive* iff the only occurrences of $\forall$, $\supset$ and $\neg$ are strictly positive, i.e. not within the antecedent of an implication or negation.

Proposition [RD & SN]. A formula is weakly positive iff the only occurrences of $\forall$, $\supset$ and $\neg$ are positive, i.e. within the scope of an even number of implications and negations.

Examples. Positive formulae (i.e. having **no** such occurrences); SCIs; Negation normal formulae; the McKinsey condition; the condition for Kreisel-Putnam logic.

New Coherentisation Algorithm: Weak Positivisation

Every first-order formula can be converted to a classically equivalent weakly positive formula as follows. Note that conversion to NNF would suffice, but may change too much of the formula.

First, we treat negations as implications. Second, if the formula is an implication with positive antecedent we leave the antecedent unchanged (and, recursively, we convert the succedent). Then (with x not free in B) we can (but don't always have to) use the classical equivalences

$$(C \supset \bot) \supset B \equiv C \vee B \quad (1)$$

$$(C \supset D) \supset B \equiv (C \wedge \neg D) \vee B \quad (2)$$

$$\forall x A \supset B \equiv \exists x.A \supset B \quad (3)$$

and the intuitionistic equivalences

$$(C \vee D) \supset B \equiv (C \supset B) \wedge (D \supset B) \quad (4)$$

$$(C \wedge D) \supset B \equiv C \supset (D \supset B) \quad (5)$$

$$\exists x A \supset B \equiv \forall x.A \supset B \quad (6)$$

New Coherentisation Algorithm: Analysis of W.P. Formula

Proposition [RD & SN]. Every weakly positive formula A **is** either

- an atom or
- a universally quantified implication $\forall \mathbf{x}.\ P \supset D$ (with P positive) and D a disjunction of zero or more existentially quantified conjunctions of zero or more of the following:
  - atoms
  - weakly positive formulae A_i simpler than A.

We allow empty quantification; we consider negations to be implications and D to be the same as $\top \supset D$.

Proof. By analysis of the structure of A. We also allow trivial disjunctions (at most one disjunct) and trivial conjunctions (at most one conjunct)—but, to avoid an infinite recursion, some step of analysing A, if non-atomic, must be non-trivial. □

New Coherentisation Algorithm: Constructing CIs

Corollary [RD & SN]. Given a weakly positive formula A, we can introduce fresh relation symbols and "semi-definitional implications" so that A is simplified to a CI and the new implications are CIs, making a coherent theory conservative over A.

Proof. By induction on the structure of A. If, when analysing a conjunction, we meet a subformula $C(\mathbf{x})$ other than an an atom or a conjunction $A_1 \wedge A_2$, we introduce in its place a fresh relation symbol N_i (with appropriate arguments) and a "semi-definitional implication" $\forall \mathbf{x}.\, N_i(\mathbf{x}) \supset C(\mathbf{x})$ (which we may need to analyse further).

Any universal quantifiers or implications (with positive antecedents) at the front of $C(\mathbf{x})$ can easily be shifted, so we get a formula of the form $\forall \mathbf{xy}.\, N_i(\mathbf{x}) \wedge P(\mathbf{x}, \mathbf{y}) \supset B(\mathbf{x}, \mathbf{y})$, where $P(\mathbf{x}, \mathbf{y})$ is positive and $B(\mathbf{x}, \mathbf{y})$ is weakly positive and smaller than A.

After a finite number of steps we have replaced A by a finite number of CIs axiomatising a theory conservative over A. □

New Coherentisation Algorithm: Constructing SCIs

We can (intuitionistically) convert the coherent implications (the CIs) to SCIs by methods discussed earlier.

We can reduce the number of steps that we go round the loop (and thus reduce the number of fresh relation symbols and of SCIs generated) by applying (as transformations) any of the intuitionistic “permutations”:

```
exists x. A v B      ==>  exists x A   v  exists x B
A & (B v C)          ==>  (A & B) v (A & C)
(B v C) & A          ==>  (B & A) v (C & A)
C & exists x D       ==>  exists x. C & D  (x chosen not free in C)
(exists x C) & D     ==>  exists x. C & D  (x chosen not free in D)
A => forall x B      ==>  forall x. A => B (x chosen not free in A)
A => B => C          ==>  (A & B) => C
~A                   ==>  A => \bot
```

Theorem (RD & SN) With or without these permutations, we have an **idempotent** translation, i.e. any SCI is transformed by this process to itself. (NB: no part of an SCI matches the LHS of any of these permutations. We allow trivial simplifications, e.g. $\bot \vee B \equiv B$.)

Remark Our examples of the McKinsey condition and that for K-P logic are translated exactly as we have illustrated. [Otherwise, trouble!]

A Glivenko-style Theorem

Theorem If $\mathcal{T}$ is a theory whose axioms are weakly positive sentences, and A is a positive sentence provable in $\mathcal{T}$, then A has an intuitionistic proof from $\mathcal{T}$.

This is just an extension of what we have called "Barr's theorem"; a proof of $\mathcal{T} \Rightarrow A$ in **G3c** is (because of the syntactic restrictions) already a proof in **m-G3i** (multi-succedent intuitionistic calculus).

Result can be strengthened by allowing A to be an SCI.

Result is due to Negri: see her paper "From rule systems to systems of rules" in the JLC 2014 on "generalised geometric implications".

Implementation

The new coherentisation algorithm is implemented in *OCaml* and available on my website. It requires the loading of John Harrison's excellent *OCaml* library for manipulating first-order terms, as documented in his wonderful book.

I don't have a *Coq* or *Isabelle* proof of its correctness. (There isn't even one for Harrison's code.)

Work is in progress on **YAPE**, Yet Another Proof Engine, in *Prolog*, which allows natural expression of rules for root-first search in sequent calculi for propositional logics such as **Int**, and LaTeX output of proofs. The 2014 version is on my website, but doesn't properly cover labelled calculi; the 2015 version (to be completed REAL SOON) will be, once termination conditions are properly implemented.

Once that is done, incorporation of ideas from coherent logic automation can be started. The two kinds of system need to interact; it's not just a matter of using the work of (e.g.) Bezem & Coquand as an oracle. Termination will of course continue to be an issue.

Opportunities and Challenges

Better automation: Polonsky's thesis (2010) (building on B & C 2005) and de Nivelle (. . . , 2014) is the state of the art. Relevant work also by Giese et al in Oslo.

Better automation, 2: Our interest however is not in repeating the work of Bezem & Coquand, or of Polonsky, et al, but in automating better the reasoning in what is in effect a mix of classical zero-order logic (apart from $\rightarrow$ or the modal operators . . .) and (for the frame conditions) coherent logic. Something different is needed: not just establishing a formula's validity, or inconsistency, or just of generating atomic consequences, as in "The Chase" algorithm, but interleaving such generation and the creation of new labels and atoms.

Better conversion of first-order formulae: again, Polonsky. Lots of opportunities for optimisation and tricks. For example, in $\forall \mathbf{x}.\ C \supset (P \vee \neg Q)$ one would do better to construct $\forall \mathbf{x}.\ (C \wedge Q) \supset P$ than to introduce a new symbol for $\neg Q$. Likewise, $\forall \mathbf{x}.\ C \supset (P \wedge \neg Q)$ splits into $\forall \mathbf{x}.\ C \supset P$ and $\forall \mathbf{x}.\ C \wedge Q \supset \bot$. Not so easy in presence of $\exists$.

Summary

So, what have we done? We have

1. tried to stick to our (unstated) principles of (i) avoiding pre-processing and (ii) retaining naturality
2. not much discussed efficiency: see work of Bezem et al.
3. retrieved from obscurity a result that uses a technique of Skolem (1920), "Relational Skolemisation" (a better name than e.g. "Atomisation"): a result best formulated as "Every f.-o. theory has a coherent conservative [relational] extension".
4. shown how to use this in adjusting f.o. frame conditions for modal and intermediate logics, allowing them to be expressed as SCIs.
5. introduced the notion of "weakly positive formula", and shown that any f.-o. formula can easily be put (equivalently) into this form, in a way that leaves w.p. formulae unchanged.
6. given (and implemented) an idempotent algorithm for converting weakly positive sentences to conjunctions of SCIs—not necessarily equivalent but at least giving a conservative extension.
7. continued implementation of generic framework for exploiting coherent axioms for intermediate and modal logics.

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